Problems. Differential calculus

1. Calculate limits of the function: $\lim_{x\to -2+} \frac{x}{x+2}$ and $\lim_{x\to\infty} \frac{x}{x+2}$.

2. Calculate limit of the function: $\lim_{x\to 2} \frac{x^2-4}{x^2-3x+2}$.

3. Calculate limit of the function: $\lim_{x\to\infty} \frac{x-1}{x^2+x+1}$.

4. Calculate limit of the function: $\lim_{x\to\infty} e^{\frac{1}{x}}$.

5. Calculate limit of the function: $\lim_{x\to\infty} \arcsin \frac{x}{x+1}$.

6. Calculate limit of the function: $\lim_{x\to 0} (1+2x)^{\frac{3}{x}}$.

7. Calculate limit of the function: $\lim_{x\to 0} \frac{\sin 3x}{\sin 2x}$.

8. Calculate limit of the function: $\lim_{x\to 0} \frac{\sin(\sin x)}{x}$.

9. Calculate limit of the function: $\lim_{x\to \frac{\pi}{2}+} e^{\operatorname{tg} x}$.

10. Calculate limit of the function: $\lim_{x\to\infty} x\left(e^{\frac{1}{x}}-1\right)$.

11. Differentiate the function: $f(x) = x^3 - 2x^2 + 4$ in $a = 2$ and write down equation for the tangent and normal to the curve in point: [2, 4].

12. Differentiate the function: $f(x) = \frac{x^5}{5} + \frac{8}{x} - \sqrt[3]{x^4} - \frac{4}{\sqrt{x}} + 5$.

13. Differentiate the function: $f(x) = \sqrt{1+3x^2}$.

14. Differentiate the function: $f(x) = \sin x \cdot \cos x - 2\operatorname{tg} x - 3$.

15. Differentiate the function: $f(x) = \frac{1-2x^3}{(1+3x)^2}$.

16. Differentiate the function: $f(x) = (x^2+1) \cdot e^x$.

17. Differentiate the function: $f(x) = \sqrt{x \cdot \ln x}$.

18. Differentiate the function: $f(x) = e^{5x^2-3x-1}$.

19. Differentiate the function: $f(x) = (\sin x)^{\cos x}$.

20. Differentiate the function: $f(x) = \ln(\sin \frac{1}{\sqrt{x}})$.

21. Differentiate the function: $f(x) = x \cdot e^{-x^2}$ v bode $x = 1$.

22. Differentiate the function: $f(x) = \arcsin(x^2 - 1)$ v bode $x = \frac{1}{2}$.

23. Calculate limit of the function using L'Hospital rule: $\lim_{x\to 0+}(\sqrt{x} \cdot \ln x)$.

24. Calculate limit of the function using L'Hospital rule: $\lim_{x\to 0+}\frac{e^{x^2}-1}{x\cdot \sin x}$.

25. Calculate limit of the function using L'Hospital rule: $\lim_{x\to 1} x^{\frac{2}{1-x}}$.

26. Calculate the increment of function: $f(x) = \frac{\sqrt{2-x}}{\sqrt{3+x}}$ in $a = 1$ for the increment $\Delta x = -0{,}04$.

27. Calculate approximate value of $\sin 43^\circ$ using differential.

28. Find Taylor polynomial of 5th degree with a center in $a = \frac{\pi}{2}$ for the function: $f(x) = \cos 2x$.

29. Calculate approximate value of $\sqrt[3]{29}$ with help of Taylor polynomial in a suitable center with the precision of 4 decimals.

30. Analyze course of the function: $f(x) = x\sqrt{x+3}$.

31. Analyze course of the function: $f(x) = (1+x)^2 e^{-x^2}$.

32. Analyze course of the function: $f(x) = \frac{e^{-x}}{x}$.

33. Analyze course of the function: $f(x) = x - 2\arctan x$.

Solutions

1. Graph of rational function $f(x) = \frac{x}{x+2}$ is a hyperbola, which is not defined in the point $x = -2$, where f is discontinuous. Right limit of function: $\lim_{x\to-2+}\frac{x}{x+2}$ can be calculated as:

$$\lim_{x\to-2+}\frac{x}{x+2} = \lim_{x\to-2+}\left(\frac{x+2-2}{x+2}\right) = \lim_{x\to-2+}\left(1-\frac{2}{x+2}\right) = 1-\lim_{x\to-2+}\frac{2}{x+2} =$$

$$= 1-\frac{2}{0+} = 1-\infty = -\infty$$

For the argument $x\to\pm\infty$ the fraction $\frac{x}{x+2}$ approaches the value of 1:

$$\lim_{x\to\infty}\frac{x}{x+2} = \lim_{x\to\infty}\frac{1}{1+\frac{2}{x}} = \frac{1}{1+0} = 1$$

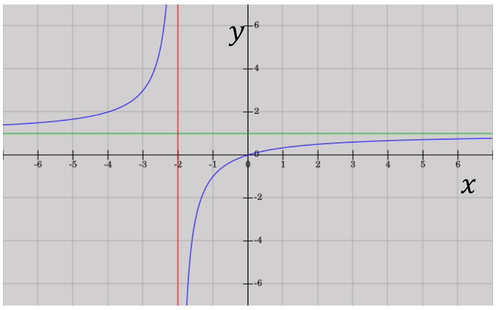

Fig. 1.2. Graph of function $f(x) = \frac{x}{x+2}$

2. $\lim_{x\to2}\frac{x^2-4}{x^2-3x+2} = \lim_{x\to2}\frac{(x-2)(x+2)}{(x-2)(x-1)} = \lim_{x\to2}\frac{x+2}{x-1} = \frac{4}{1} = 4.$

3. $\lim_{x\to\infty}\frac{x-1}{x^2+x+1} = \lim_{x\to\infty}\frac{\frac{1}{x}-\frac{1}{x^2}}{1+\frac{1}{x}+\frac{1}{x^2}} = \frac{0-0}{1+0+0} = 0.$

4. $\lim_{x\to\infty}e^{\frac{1}{x}} = e^{\lim_{x\to\infty}\frac{1}{x}} = e^0 = 1.$

5. $\lim_{x\to\infty}\arcsin\frac{x}{x+1} = \arcsin(\lim_{x\to\infty}\frac{x}{x+1}) = \arcsin(\lim_{x\to\infty}\frac{1}{1+\frac{1}{x}}) = \arcsin 1 = \frac{\pi}{2}.$

6. $\lim_{x\to0}(1+2x)^{\frac{3}{x}} = \lim_{x\to0}(1+2x)^{\frac{1}{2x}\cdot6} = e^6,$ because $\lim_{z\to0}(1+z)^{\frac{1}{z}} = e.$

7. $\lim_{x\to0}\frac{\sin3x}{\sin2x} = \lim_{x\to0}\frac{3x\cdot\frac{\sin3x}{3x}}{2x\cdot\frac{\sin2x}{2x}} = \lim_{x\to0}\frac{3x}{2x} = \frac{3}{2},$ because $\lim_{z\to0}\frac{\sin z}{z} = 1.$

8. $\lim_{x\to0}\frac{\sin(\sin x)}{x} = \lim_{x\to0}\frac{\sin(\sin x)}{\sin x}\cdot\frac{\sin x}{x} = \lim_{x\to0}\frac{\sin(\sin x)}{\sin x}\cdot\lim_{x\to0}\frac{\sin x}{x} \rightrightarrows$ substitution

$$\rightrightarrows\begin{bmatrix}\sin x = z\\ x\to0\\ z\to0\end{bmatrix} = \lim_{z\to0}\frac{\sin z}{z}\cdot\lim_{x\to0}\frac{\sin x}{x} = 1\cdot1 = 1.$$

9. $\lim_{x\to\frac{\pi}{2}^+} e^{\operatorname{tg} x} = e^{\lim_{x\to\frac{\pi}{2}^+} \operatorname{tg} x} = e^{-\infty} = \frac{1}{\infty} = 0.$

10. $\lim_{x\to\infty} x\left(e^{\frac{1}{x}} - 1\right) \rightrightarrows \begin{bmatrix} e^{\frac{1}{x}} - 1 = t & x = \frac{1}{\ln(t+1)} \\ x\to\infty & t\to 0 \end{bmatrix} = \lim_{t\to 0} \frac{t}{\ln(t+1)} = \lim_{t\to 0} \frac{1}{\frac{1}{t}\cdot\ln(1+t)} =$

$= \lim_{t\to 0} \frac{1}{\ln(1+t)^{\frac{1}{t}}} = \frac{1}{\ln\left[\lim_{t\to 0}(1+t)^{\frac{1}{t}}\right]} = \frac{1}{\ln e} = 1.$

11. Derivative of function: $f(x) = x^3 - 2x^2 + 4$ in the point $a = 2$ and equations of a tangent and normal to the curve in point $[a, f(a)] = [2, 4]$ can be found as:
Derivative: $f'(x) = 3x^2 - 4x$, slope of tangent k_t in point bode $a = 2$ will be $k_t = f'(2) = 3\cdot 2^2 - 4\cdot 2 = 4$. Equation of the tangent will be:

$$y - f(a) = k_t(x - a)$$
$$y - 4 = 4(x - 2)$$
$$y = 4x - 4$$

Slope of normal k_n in point $a = 2$ will be: $k_n = -\frac{1}{f'(x)} = -\frac{1}{4}$. Equation of the tangent will have the shape: $y - f(a) = k_n(x - a)$

$$y - 4 = -\frac{1}{4}(x - 2)$$
$$y = -\frac{1}{4}x + \frac{9}{2}$$

12. Derivative of function: $f(x) = \frac{x^5}{5} + \frac{8}{x} - \sqrt[3]{x^4} - \frac{4}{\sqrt{x}} + 5$ is calculated as:

$\left(\frac{x^5}{5} + \frac{8}{x} - \sqrt[3]{x^4} - \frac{4}{\sqrt{x}} + 5\right)' = \frac{1}{5}5x^4 + 8(-1)x^{-2} - \frac{4}{3}x^{\frac{1}{3}} - 4\left(-\frac{1}{2}\right)x^{-\frac{3}{2}} + 0 =$

$= x^4 - \frac{8}{x^2} - \frac{4}{3}\sqrt[3]{x} + \frac{2}{\sqrt{x^3}}.$

13. $(\sqrt{1 + 3x^2})' = \frac{1}{2}(1 + 3x^2)^{-\frac{1}{2}}\cdot 6x = \frac{3x}{\sqrt{1+3x^2}}.$

14. $(\sin x\cdot\cos x - 2\tan x - 3)' = (\sin x\cdot\cos x - 2\frac{\sin x}{\cos x} - 3)' =$

$= \cos x\cdot\cos x + \sin x\cdot(-\sin x) - 2\frac{\cos x\cdot\cos x - \sin x\cdot(-\sin x)}{\cos^2 x} - 0 =$

$= \cos^2 x - \sin^2 x - 2\frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \cos^2 x - \sin^2 x - 2\tan^2 x - 2$

15. $(\frac{1-2x^3}{(1+3x)^2})' = \frac{-6x^2(1+3x)^2 - (1-2x^3)2(1+3x)3}{(1+3x)^4} = \frac{-18x^4 - 24x^3 - 6x^2 - 18x - 6}{(1+3x)^4} = \frac{-6(3x^4 + 4x^3 + x^2 + 3x + 1)}{(1+3x)^4}.$

16. $[(x^2 + 1)\cdot e^x]' = 2x\cdot e^x + (x^2 + 1)\cdot e^x = (x + 1)^2\cdot e^x.$

17. $(\sqrt{x\cdot\ln x})' = \frac{1}{2}(x\cdot\ln x)^{-\frac{1}{2}}\cdot\left(\ln x + x\frac{1}{x}\right) = \frac{\ln x + 1}{2\sqrt{x\cdot\ln x}}.$

18. $(e^{5x^2 - 3x - 1})' = e^{5x^2 - 3x - 1}\cdot(10x - 3).$

19. $[(\sin x)^{\cos x}]' = [e^{\cos x \cdot \ln(\sin x)}]' =$

$$= e^{\cos x \cdot \ln(\sin x)}\left[(-\sin x)\cdot \ln(\sin x) + \cos x \cdot \frac{\cos x}{\sin x}\right] =$$

$$= (\sin x)^{\cos x}\left[\frac{\cos^2 x}{\sin x} - \sin x \cdot \ln(\sin x)\right].$$

20. $\left[\ln\left(\sin\frac{1}{\sqrt{x}}\right)\right]' = \frac{1}{\sin\frac{1}{\sqrt{x}}}\cdot \cos\frac{1}{\sqrt{x}}\cdot\left(-\frac{1}{2}x^{-\frac{3}{2}}\right) = -\frac{1}{2\sqrt{x^3}}\cdot \operatorname{cotg}\frac{1}{\sqrt{x}}.$

21. Second derivative of function: $f(x) = x\cdot e^{-x^2}$ in point $x = 1$, can be calculated by differentiating two times and inserting $x = 1$:

$(x\cdot e^{-x^2})' = e^{-x^2} + x\cdot e^{-x^2}(-2x) = (1-2x^2)\cdot e^{-x^2}$

$(x\cdot e^{-x^2})'' = ((1-2x^2)\cdot e^{-x^2})' = -4x\cdot e^{-x^2} + (1-2x^2)\cdot e^{-x^2}(-2x) =$

$= (4x^3 - 6x)e^{-x^2}$

$f''(1) = (4\cdot 1^3 - 6\cdot 1)\, e^{-1^2} = -\frac{2}{e}.$

22. $[\arcsin(x^2-1)]' = \frac{1}{\sqrt{1-(x^2-1)}}\cdot 2x = \frac{2x}{\sqrt{2-x^2}}$

$[\arcsin(x^2-1)]'' = \left(\frac{2x}{\sqrt{2-x^2}}\right)' = \frac{2\cdot\sqrt{2-x^2} - 2x\cdot\frac{1}{2\sqrt{2-x^2}}\cdot(-2x)}{2-x^2} = \frac{2\sqrt{2-x^2}+\frac{2x^2}{\sqrt{2-x^2}}}{2-x^2} = \frac{4}{\sqrt{(2-x^2)^3}}$

$f''\left(\frac{1}{2}\right) = \frac{4}{\sqrt{\left(2-\left(\frac{1}{2}\right)^2\right)^3}} = \frac{32}{7\sqrt{7}}.$

23. Limit: $\lim_{x\to 0+}(\sqrt{x}\cdot \ln x)$ is calculated by the L'Hospital rule:

$\lim_{x\to 0+}(\sqrt{x}\cdot\ln x) = \lim_{x\to 0+}\frac{\ln x}{\frac{1}{\sqrt{x}}} = \lim_{x\to 0+}\frac{\frac{1}{x}}{-\frac{1}{2\sqrt{x^3}}} = -2\cdot\lim_{x\to 0+}\sqrt{x} = -2\cdot 0 = 0.$

24. $\lim_{x\to 0+}\frac{e^{x^2}-1}{x\cdot\sin x} = \lim_{x\to 0+}\frac{e^{x^2}\cdot 2x}{\sin x + x\cdot\cos x} = \lim_{x\to 0+}\frac{2e^{x^2}+2x\cdot e^{x^2}2x}{\cos x+\cos x - x\cdot\sin x} = \lim_{x\to 0+}\frac{2e^{x^2}+4x^2e^{x^2}}{2\cos x - x\cdot\sin x} =$

$= \frac{2\cdot 1+0}{2\cdot 1-0} = 1.$

25. $\lim_{x\to 1} x^{\frac{2}{1-x}} = \lim_{x\to 1} e^{\frac{2}{1-x}\cdot\ln x} = e^{\lim_{x\to 1}\frac{2\ln x}{1-x}} = e^{\lim_{x\to 1}\frac{\frac{2}{x}}{-1}} = e^{\lim_{x\to 1}\frac{-2}{x}} = e^{-2} = \frac{1}{e^2}.$

26. Differential $\Delta f(x)$ of function $f(x) = \frac{\sqrt{2-x}}{\sqrt{3+x}}$ in the number $a = 1$ for increment $\Delta x = -0{,}04$ can be calculated as: $\Delta y \approx f'(a)\cdot\Delta x$. First we differentiate the function:

$f'(x) = \left(\frac{\sqrt{2-x}}{\sqrt{3+x}}\right)' = \frac{-\frac{1}{2\sqrt{2-x}}\cdot\sqrt{3+x}+\sqrt{2-x}\cdot\frac{1}{2\sqrt{3+x}}}{3+x} = \frac{-\frac{1\sqrt{3+x}}{2\sqrt{2-x}}+\frac{1\sqrt{2-x}}{2\sqrt{3+x}}}{3+x}.$

$f'(1) = \frac{-\frac{1\sqrt{3+1}}{2\sqrt{2-1}}+\frac{1\sqrt{2-1}}{2\sqrt{3+1}}}{3+1} = \frac{-1+\frac{1}{2}\frac{1}{2}}{4} = -\frac{3}{16}$

Differential in $a = 1$ for $\Delta x = -0{,}04$ will be:

$\Delta y \approx f'(a)\cdot\Delta x = -\frac{3}{16}\cdot\left(-\frac{4}{100}\right) = \frac{3}{400} = 0{,}0075.$

27. Calculation of approximate value of $\sin 43°$ will be carried out by means of differential $f(x) \approx (a) + f'(a) \cdot \Delta x$

We use known value of $\sin 45° = \sin\frac{\pi}{4} = \frac{\sqrt{2}}{2}$ in a nearby point $a = 45° = \frac{\pi}{4}$, derivative of the function sine $(\sin a)' = \cos a = \cos\frac{\pi}{4} = \frac{\sqrt{2}}{2}$ and increment of the variable $\Delta x = x - a = 43° - 45° = -2° = -\frac{\pi}{90}$. Thus:

$$\sin 43 \approx \sin\frac{\pi}{4} + \cos\frac{\pi}{4} \cdot \left(-\frac{\pi}{90}\right) = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} \cdot \left(-\frac{\pi}{90}\right) = \frac{\sqrt{2}}{2} \cdot \frac{90-\pi}{90} \approx 0{,}6824$$

For comparison, the value of $\sin 43°$ obtained by a calculator rounded of to 4 decimals is equal to 0,6820.

28. Taylor polynomial of 5th degree with a center in $a = \frac{\pi}{2}$ for function $f(x) = \cos 2x$ will be:

$$T_{5,f,\frac{\pi}{2}}(x) = f\left(\frac{\pi}{2}\right) + \frac{f'\left(\frac{\pi}{2}\right)}{1!}\left(x - \frac{\pi}{2}\right) + \frac{f''\left(\frac{\pi}{2}\right)}{2!}\left(x - \frac{\pi}{2}\right)^2 + \ldots + \frac{f^{(5)}\left(\frac{\pi}{2}\right)}{5!}\left(x - \frac{\pi}{2}\right)^5$$

First we calculate derivative of f in point $x = \frac{\pi}{2}$ $(2x = \pi)$:

$f(x) = \cos 2x$	$f\left(\frac{\pi}{2}\right) = -1$
$f'(x) = -2\sin 2x$	$f'\left(\frac{\pi}{2}\right) = 0$
$f''(x) = -4\cos 2x$	$f''\left(\frac{\pi}{2}\right) = 4$
$f'''(x) = 8\sin 2x$	$f''\left(\frac{\pi}{2}\right) = 0$
$f^{(4)}(x) = 16\cos 2x$	$f^{(4)}\left(\frac{\pi}{2}\right) = -16$
$f^{(5)}(x) = -32\sin 2x$	$f^{(5)}\left(\frac{\pi}{2}\right) = 0$

The derivatives are inserted into Taylor polynomial:

$$T_{5,f,\frac{\pi}{2}}(x) = -1 + \frac{0}{1!}\left(x - \frac{\pi}{2}\right) + \frac{4}{2!}\left(x - \frac{\pi}{2}\right)^2 + \frac{0}{3!}\left(x - \frac{\pi}{2}\right)^3 - \frac{16}{4!}\left(x - \frac{\pi}{2}\right)^4 + \frac{0}{5!}\left(x - \frac{\pi}{2}\right)^5 =$$

$$= -1 + 2\left(x - \frac{\pi}{2}\right)^2 - \frac{2}{3}\left(x - \frac{\pi}{2}\right)^4.$$

29. The value of $\sqrt[3]{29}$ with a precion to 4 decimals will be calculated by Taylor polynomial for the function $f(x) = \sqrt[3]{x}$ with a center close to number 29 i.e. in $x = 27$ (which is conveniently selected since $\sqrt[3]{27} = 3$, $\Delta x = 2$). The required degree of the polynomial will be determined by calculating difference between approximate and 'precise' value of $\sqrt[3]{29}$.

First we calculate the derivatives of f in $x = 27$:

$f(x) = \sqrt[3]{x}$	$f(27) = 3$
$f'(x) = \frac{1}{3} \cdot \frac{1}{\sqrt[3]{x^2}}$	$f'(27) = \frac{1}{27}$
$f''(x) = -\frac{2}{9} \cdot \frac{1}{\sqrt[3]{x^5}}$	$f''(27) = -\frac{2}{2187}$
$f'''(x) = \frac{10}{27} \cdot \frac{1}{\sqrt[3]{x^8}}$	$f'''(27) = \frac{10}{177147}$

$f^{(4)}(x) = -\frac{80}{81} \cdot \frac{1}{\sqrt[3]{x^8}}$ $\quad$ $f^{(4)}(27) = -\frac{80}{14348907}$

Then we inserts the expressions into Taylor polynomial:

$$T_{4,f,27}(29) = 3 + \frac{1}{27 \cdot 1!} \cdot 2 - \frac{2}{2187 \cdot 2!} \cdot 2^2 + \frac{10}{177147 \cdot 3!} \cdot 2^3 - \frac{80}{14348907 \cdot 4!} \cdot 2^4 \cong$$
$$\cong 3 + 0{,}074074 - 0{,}001829 + 0{,}000075 - 0{,}0000037 = 3{,}072316$$

Using calculator we obtain $\sqrt[3]{29} = 3{,}072317$. This number corresponds to the approximately estimated value up to 5 decimals. To estimate the $\sqrt[3]{29}$ with a precision of 4 decimals it is sufficient to take into account first 3 members of the Taylor expansion.

30. During analysis of the course of function $f(x) = x\sqrt{x+3}$ we will follow the scheme shown below:

- domain, periodicity, even/odd behavior,
- signs/values, roots, axis cross points,
- limits in marginal points of D_f,
- $f'(x)$: regions of increase/decrease,
- $f''(x)$: regions of concavity, inflection points, local extremes,
- global maxima/minima,
- asymptotes,
- graph

Domain can be determined form condition: $x + 3 \geq 0$. We obtain: $D_f = \langle -3, \infty)$. Function is not even nor odd as: $f(x) \neq f(-x)$ and $f(x) \neq -f(-x)$. The curve crosses the coordinate axes in points: $[-3, 0]$ and $[0, 0]$. for $x \to \infty$ the limit $\lim_{x\to\infty} x\sqrt{x+3} = \infty$ and the function is increasing without bounds. First derivative: $f'(x) = \sqrt{x+3} + \frac{x}{2\sqrt{x+3}} = \frac{3(x+2)}{2\sqrt{x+3}}$. Function is increasing in the region of D_f, where $x + 2 > 0$, i.e. $x > -2$, namely on interval $(-2, \infty)$ and decreasing where: $x < -2$, i.e. on interval $\langle -3, -2)$. Function has stationary point $f'(x) = 0$ for $3(x+2) = 0$, i.e. $x = -2$. Second derivative of f is equal to: $f''(x) = \frac{3 \cdot 2\sqrt{x+3} - 3(x+2) \cdot \frac{1}{\sqrt{x+3}}}{4(x+3)} = \frac{3(x+4)}{8\sqrt{(x+3)^3}}$. Second derivative is positive for: $x + 4 > 0$, i.e. for $x > -4$, namely on the whole D_f. Therefore, the function is concave upwards on the whole D_f. $f''(-2) = \frac{3}{2} > 0$, therefore, the function has a local as well as global minima in the point $[-2, -2]$. The function does not have any points of inflection $f''(x) \neq 0$, nor local maxima. It does not have aby asymptotes. Graph of the function f is shown on Fig. 1.3.

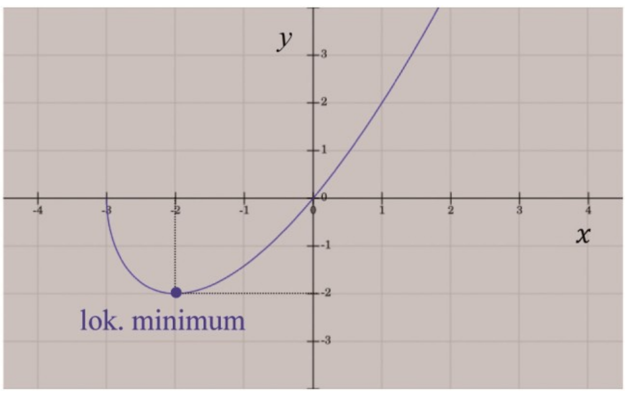

Fig. 1.3. Graph of function $f(x) = x\sqrt{x+3}$

31. During the analysis of the course of function $f(x) = (1+x^2)e^{-x^2}$ we will follow the same scheme as in the previous problem.
Domain of the function is: $D_f = (-\infty, \infty)$. Function is even, because: $f(x) = f(-x)$. The curve crosses the y axis in the point $[0, 1]$, does not cross the x axis. In the marginal points of D_f: $\lim_{x\to\pm\infty}(1+x^2)e^{-x^2} = \lim_{x\to\pm\infty}\frac{1}{e^{x^2}} = \frac{1}{\pm\infty} = 0$, thus the function is approaching the x axis for $x \to \pm\infty$. First derivative: $f'(x) = [(1+x^2)e^{-x^2}]' = \frac{2x\cdot e^{x^2} - (1+x^2)\cdot e^{x^2}\cdot 2x}{e^{2x^2}} = -\frac{2x^3}{e^{x^2}}$. Function is increasing on interval: $x \in (-\infty,\ 0)$, and decreasing on $x \in (0, \infty)$. Function has a stationary point in pre $x = 0$, it is a local and also a global minima with coordinates $[0,\ 1]$. Second derivative of f is: $f''(x) = [-\frac{2x^3}{e^{x^2}}]' = \frac{4x^4 - 6x^2}{e^{x^2}} = \frac{2x^2}{e^{x^2}}(\sqrt{2}x - \sqrt{3})(\sqrt{2}x + \sqrt{3})$. The second derivative is positive and f is concave upwards on: $x \in \left(-\infty, -\sqrt{\frac{3}{2}}\right) \cup \left(\sqrt{\frac{3}{2}}, \infty\right)$. The function is concave downwards on: $x \in \left(-\sqrt{\frac{3}{2}}, \sqrt{\frac{3}{2}}\right)$. The function has two points of inflection in $x = \pm\sqrt{\frac{3}{2}}$.
In infinite points of D_f the function is asymptoticaly approaching the x axis ($y = 0$). The graph of the function is shown below.

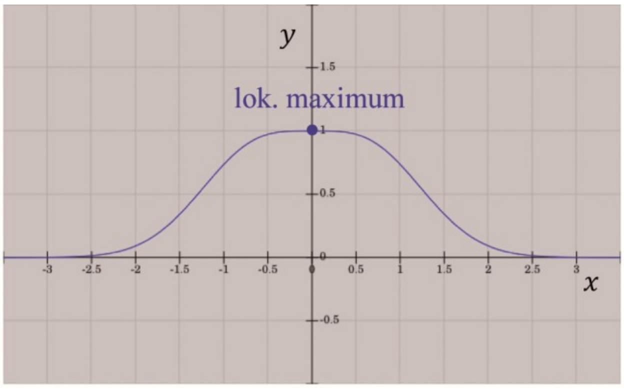

Fig. 1.4. Graph of function $f(x) = (1+x^2)e^{-x^2}$

32. Domain of the function $f(x) = \frac{e^{-x}}{x}$ is: $D_f = (-\infty, 0) \cup (0, \infty)$. Function is not even nor odd. It does not cross the x axis. In the marginal points of D_f: for $x \to \infty$ $\lim_{x\to\infty}\frac{e^{-x}}{x} = -\lim_{x\to\infty}\frac{e^{-x}}{1} = 0$, function is approaching the x axis, for $x \to -\infty$ $\lim_{x\to-\infty}\frac{e^{-x}}{x} = -\lim_{x\to-\infty}\frac{e^{-x}}{1} = -\infty$, the function is decreasing. The function has different right and left limits in the point $x = 0$: $\lim_{x\to 0-}\frac{e^{-x}}{x} = \lim_{x\to 0-}\frac{1}{x\cdot e^{x}} = \frac{1}{0-} = -\infty$. a $\lim_{x\to 0+}\frac{e^{-x}}{x} = \lim_{x\to 0+}\frac{1}{x\cdot e^{x}} = \frac{1}{0+} = +\infty$. Therefore the function has an asymptote without slope $x = 0$.

First derivative: $f'(x) = (\frac{e^{-x}}{x})' = \frac{-e^{-x} \cdot x - e^{-x} \cdot 1}{x^2} = \frac{-(1+x)e^{-x}}{x^2}$. The function has a stationary point in: $x = -1$. The function is decreasing for $x > -1$ and increasing for $x < -1$. Second derivative f is equal to: $f''(x) = \frac{(e^{-x}(x+1) - e^{-x})x^2 + e^{-x}(x+1)2x}{x^4} = \frac{e^{-x}(x^2+2x+2)}{x^3}$. In a stationary point $x = -1$the function has a local maxima with coordinates $[-1, -e]$, because $f''(-1) = -e < 0$. Second derivative is negative and the function is concave downward for $x \in (-\infty, 0)$ and concave upward on $x \in (0, \ \infty)$. The function does not have points of inflection. Graph of the function is shown below.

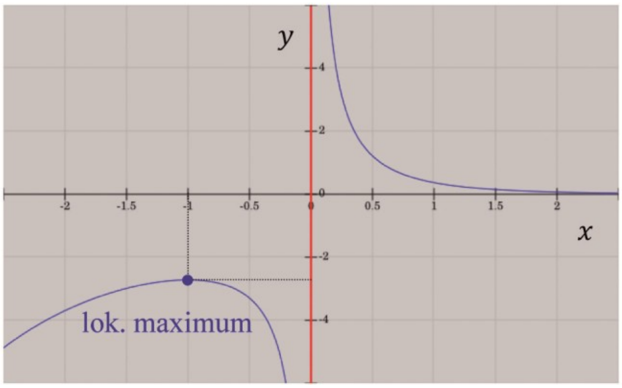

Fig. 1.5. Graph of function $f(x) = \frac{e^{-x}}{x}$

33. Domain of the function $f(x) = x - 2\arctan x$ is $D_f = (-\infty, \infty)$. The function is odd. It crosses the x axis in points: $x = -\frac{3}{4}\pi, \ 0, \ \frac{3}{4}\pi$. In the marginal points of D_f for $x \to \pm\infty$ $\lim_{x \to \pm\infty}(x - 2\arctan x) = \pm\infty \pm \frac{\pi}{2} = \pm\infty$ the function is increasing/decreasing. The function does not have asymptotes without slope, but has two asymptotes with a slope: $k = \lim_{x \to \pm\infty} \frac{x - 2\arctan x}{x} = 1 - \lim_{x \to \pm\infty} \frac{2\arctan x}{x} = 1 - \frac{\pi}{\pm\infty} = 1 \pm 0 = 1$, $q = \lim_{x \to \pm\infty}(\ x - 2\arctan x - x) = 2\lim_{x \to \pm\infty} \arctan x = \pm\pi$. Equation of the first asymptote a_1 is: $y = x - \pi$ the second a_2: $y = x + \pi$. First derivative: $f'(x) = (x - 2\arctan x)' = 1 - \frac{2}{1+x^2} = \frac{(x+1)(x-1)}{1+x^2}$. The function has stationary points for $x = \pm 1$. The function is increasing on interval $x \in (-\infty, -1) \cup (1, \infty)$ and decreasing on $x \in (-1, 1)$. Second derivative of f is: $f''(x) = \frac{4x}{(x^2+1)^2}$. In the stationary point $x = -1$, where $f''(-1) = -1 < 0$, the function has a local maxima with coordinates $[-1, \frac{\pi}{2} - 1]$ and in point $x = 1$, $f''(1) = 1 > 0$, has a local minima with the coordinates $[1, -\frac{\pi}{2} + 1]$. In point $x = 0$ there is a point of inflection since $f'''(x) = \frac{4-12x^2}{(x^2+1)^3}$ and $f'''(0) = 4 \neq 0$ for an odd order of derivative. Second derivative is positive and the curve is concave upwards on $x \in (0, \infty)$ and concave downwards on $x \in (-\infty, 0)$. Grafph of the function is depicted on the figure below.

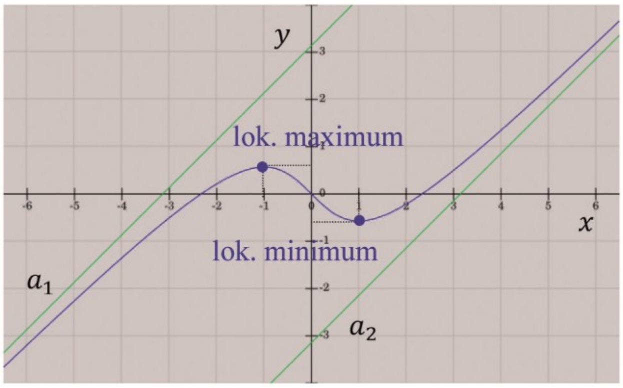

Fig. 1.6. Graph of function $f(x) = x - 2\arctan x$