Solved Problems I.

Problems. Elementary real functions

1. Find domain D_f of function $f(x) = \frac{1}{\sqrt{5-x}}$.

2. Determine whether function $f(x) = \frac{x^2}{x+3}$ is even or odd.

3. Determine whether function $f(x) = \sin(3x + 1)$ is a periodic function and find its period p.

4. Find inverse functions to the functions: $f(x) = \frac{2x-1}{3x+5}$ and $f(x) = \ln(5 - 2x)$.

5. Find composed function h, obtained by composition of real functions f a g, if: $f(x) = \log x$ and $g(x) = x^3 - 2x$.

6. Find equation of a straight line, which passes through point $[3, -1]$ and forms with axis o_x an angle of 135° $\left(\frac{3}{4}\pi\right)$.

7. Find the roots of algebraic equation: $x^4 - 2x^3 - x^2 + 2x = 0$.

8. Divide rational function into partial fractions: $R(x) = \frac{x^4+2x^3-10x^2+22x-71}{x^2+2x-15}$.

9. Solve logarithmic equation: $\log_2(9 - 2^x) = 3 - x$ where $2^x < 9$.

10. Solve trigonometric equation: $\operatorname{tg} x \cdot (\sin x - 1) = \frac{1}{2\cos x} - \cos x$, $x \neq (2k + 1)\frac{\pi}{2}$.

Solutions

1. Domain D_f of the function $f(x) = \frac{1}{\sqrt{5-x}}$, can be found as an interval $x \in \mathbb{R}$, on which the arithmetic operations division and square root are defined: $\sqrt{5-x} \neq 0$ and $5 - x \geq 0$. by solving the inequalities we obtain: $D_f = (-\infty, 5)$.

2. Function $f(x) = \frac{x^2}{x+3}$, $D_f = \mathbb{R} \setminus \{-3\}$ is not even nor odd as:

$$f(-x) = \frac{(-x)^2}{(-x)+3} = \frac{x^2}{3-x} \neq f(x) \;\wedge\; f(-x) \neq -f(x)$$

3. We look for p such that for each $x \in D_f$ of the function $f(x) = \sin(3x+1)$ holds: $f(x+p) = f(x)$. The sine function is periodic with a period of 2π, therefore: $\sin[(3x+1) + 2\pi] = \sin[3(x + \frac{2\pi}{3}) + 1]$, it follows that: $p = \frac{2\pi}{3} = 2{,}094$.

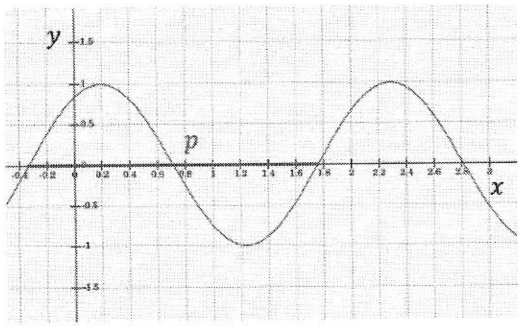

Fig. 1.1. Graph of function $f(x) = \sin(3x+1)$

4. The search for inverse function consist in exchange of the independent and dependent variables and expressing $f^{-1}(x)$:

$$x = \frac{2y-1}{3y+5} \quad \text{then} \quad y = f^{-1}(x) = \frac{5x+1}{2-3x} \quad \text{or}$$

$$x = \ln(5-2y) \quad \text{then} \quad y = f^{-1}(x) = \frac{5-e^x}{2}$$

5. Domain of function $f(x) = \log x$ is $D_f = (0, \infty)$ and $H_f = (-\infty, \infty)$. Second function $g(x) = x^3 - 2x$ has $D_g = \mathbb{R}$ and $H_g = \mathbb{R}$. Composition of functions $h = f \circ g$ on $D_{f \circ g} = (0, \infty)$ can be found by searching for function $h(x)$, which sets in by combining the 'outer' function $f(u) = \log u$ and 'internal' function $u = g(x) = x^3 - 2x$ as: $h = f \circ g = f(g(x)) = \log(x^3 - 2x)$.

6. Equation of a straight line with a slope $\tan\alpha$, which runs through a point with coordinates $[a, b]$ will have be: $y - b = \tan\alpha \cdot (x - a)$. In our case: $y - (-1) = \tan 135^o \cdot (x - 3)$. Since $\tan 135^o = -1$, after rearrangement we get the equation: $y = -x + 2$.

7. Solution of the algebraic equation: $x^4 - 2x^3 - x^2 + 2x = 0$ can be found with help of the following modifications: $x^4 - 2x^3 - x^2 + 2x = 0$

$$x^3(x-2) - x(x-2) = 0$$

$$(x-2)(x^3 - x) = 0$$

$$(x-2)x(x-1)(x+1) = 0$$

This equation is satisfied if any of the multiplicand is equal to zero:

$$x - 2 = 0 \qquad x_1 = 2$$

$$x = 0 \qquad x_2 = 0$$

$$x - 1 = 0 \qquad x_3 = 1$$

$$x + 1 = 0 \qquad x_4 = -1$$

8. Rational function: $R(x) = \frac{x^4+2x^3-10x^2+22x-71}{x^2+2x-15}$ is not purely rational, therefore we divide the numerator by the denominator:

$$(x^4 + 2x^3 - 10x^2 + 22x - 71):(x^2 + 2x - 15) = x^2 + 5 + \frac{12x+4}{x^2+2x-15}$$

Resulting function is a polynomial of second degree and a purely rational function. This can be turned into partial fractions as:

$$\frac{12x+4}{x^2+2x-15} = \frac{12x+4}{(x-3)(x+5)} = \frac{A}{x-3} + \frac{B}{x+5}$$

from which:

$$12x + 4 = A(x+5) + B(x-3)$$

$$12x + 4 = (A+B)x + 5A - 3B$$

By comparing coefficients of equal powers of x, we get:

$$12 = A + B \quad \text{and} \quad 4 = 5A - 3B$$

From these two equations we get a solution: $A = 5$ a $B = 7$

Then: $R(x) = \frac{x^4+2x^3-10x^2+22x-71}{x^2+2x-15} = x^2 + 5 + \frac{5}{x-3} + \frac{7}{x+5}$

9. Logarithmic equation can be converted to exponential one:

$$\log_2(9 - 2^x) = 3 - x$$

$$2^{3-x} = 9 - 2^x$$

$$2^3 \frac{1}{2^x} = 9 - 2^x \quad \text{substitution:} \quad 2^x = z$$

$$9 - z = \frac{8}{z}$$

$$z^2 - 9z + 8 = 0$$

$$(z-1)(z-8) = 0$$

Solution: $z = 1 \Rightarrow 2^x = 1 \Rightarrow x_1 = 0$

$z = 8 \Rightarrow 2^x = 2^3 \Rightarrow x_2 = 3$

10. Solution of this trigonometric equation: $\tan x \cdot (\sin x - 1) = \frac{1}{2\cos x} - \cos x$,
$x \neq (2k+1)\frac{\pi}{2}$ $k \in \mathbb{Z}$, can be obtained through simplifing of the equation:

$$\operatorname{tg} x \cdot (\sin x - 1) = \frac{1}{2\cos x} - \cos x$$

$$\frac{\sin^2 x}{\cos x} - \frac{\sin x}{\cos x} = \frac{1}{2\cos x} - \cos x \quad /\cdot \cos x$$

$$\sin^2 x - \sin x = \frac{1}{2} - \cos^2 x \quad \text{substitution: } \cos^2 x = 1 - \sin^2 x$$

$$\sin^2 x - \sin x = \frac{1}{2} - 1 + \sin^2 x$$

$$\sin x = \frac{1}{2}$$

Solution: $x_1 = \frac{\pi}{6} + 2k\pi$ a $x_2 = \frac{5\pi}{6} + 2k\pi, k \in \mathbb{Z}$

TOMH GROUP

Problems. Infinite sequences and power series

1. Write down first 5 members of the sequence whose n^{th} member is defined as: $a_n = \frac{n+1}{2^n}$.

2. Find the formula for the n^{th} member of the sequence: $\{b_n\} = \left\{-1, \frac{1}{4}, -\frac{1}{9}, \frac{1}{16}, -\frac{1}{25}, \dots\right\}$.

3. Calculate the sum of first 10 members of arithmetic sequence: $a_n = 2 + (n-1) \cdot 3$.

4. Calculate the sum of first 15 members of geometric sequence:

$$\{b_n\} = \{b_1 q^{n-1}\} = \left\{3\left(\frac{1}{4}\right)^{n-1}\right\}.$$

5. Prove that sequence $\left\{\frac{n-1}{n+3}\right\}_{n=1}^{\infty}$ is increasing and bounded.

6. Find limit of the infinite sequence: $\lim_{n\to\infty} \frac{4n^3-2n+5}{3n^4+n^2-2n-3}$.

7. Find limit of the infinite sequence: $\lim_{n\to\infty} \left(1+\frac{1}{5n^2}\right)^{3n^2+2n}$.

8. Find limit of the infinite sequence: $\lim_{n\to\infty} \left(1+\frac{2}{n}\right)^{\frac{n+1}{2n+3}}$.

9. Find limit of the infinite sequence: : $\lim_{n\to\infty} 4^{\frac{6n+2}{3n-4}}$.

10. Determine whether the infinite series: $\sum_{n=0}^{\infty} 2\left(-\frac{3}{5}\right)^n$ converges. If yes, then find its sum.

11. Decide with help of D'Alembert criterion, whether the infinite series: $\sum_{n=1}^{\infty} \frac{n+1}{n}\left(\frac{1}{2}\right)^n$ converges or diverges.

12. Find the interval of convergence of the power series: $\sum_{n=1}^{\infty} \frac{1}{n^2} x^n$.

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Solutions

1. First 5 members of the sequence defined as: $a_n = \frac{n+1}{2^n}$ can be obtained by inserting $n = 1, \dots, 5$ into the formula for a_n: $\{a_n\} = \left\{1, \frac{3}{4}, \frac{4}{8}, \frac{5}{16}, \frac{6}{32}, \dots\right\}$.

2. Sequence $\{b_n\} = \left\{-1, \frac{1}{4}, -\frac{1}{9}, \frac{1}{16}, -\frac{1}{25}, \dots\right\}$ defined by the list of its first 5 members display patterns repeated in the numerator and denominator of the fractions $b_n = \frac{c_n}{d_n}$, which can easily be generalized: $\{b_n\} = \left\{\frac{(-1)^n}{n^2}\right\}$.

Index	1	2	3	4	5	...	n
$b_n = \frac{c_n}{d_n}$	$\frac{-1}{1}$	$\frac{1}{4}$	$\frac{-1}{9}$	$\frac{1}{16}$	$\frac{-1}{25}$	...	$\frac{(-1)^n}{n^2}$

3. Sum of first 10 members of arithmetic sequence: $a_n = 2 + (n-1) \cdot 3$ can be calculated from the first and tenth member: $a_1 = 2$ a $a_{10} = 29$ as:

$$S_{10} = \frac{n}{2}(a_1 + a_n) = \frac{10}{2}(2 + 29) = 155$$

4. Sum of first 15 members of geometric sequence: $\{b_n\} = \{b_1 q^{n-1}\} = \left\{3\left(\frac{1}{4}\right)^{n-1}\right\}$ can be calculated as:

$$S_{15} = b_1 \frac{1-q^{15}}{1-q} = 3\frac{1-\frac{1}{4^{15}}}{1-\frac{1}{4}} = 3\frac{1-\frac{1}{1073741824}}{0{,}75} \doteq 3{,}99999$$

5. Whether sequence $\left\{\frac{n-1}{n+3}\right\}_{n=1}^{\infty}$ is growing and bounded can be determined as follows. For an increasing sequence:

$$a_n < a_{n+1}$$

$$\frac{n-1}{n+3} < \frac{(n+1)-1}{(n+1)+3}$$

$$\frac{n-1}{n+3} < \frac{n}{n+4}$$

$$(n-1)(n+4) < n(n+3)$$

$$n^2 + 3n - 4 < n^2 + 3n$$

$$-4 < 0$$

this is valid for any value of n, therefore the sequence is increasing.
For $n = 1$ we obtain the lowest value of a_n (lower bound H_s) of increasing sequence ($a_1 = 0$). Then for each $n \in \mathbb{N}$: $a_n \geq H_s = 0$. The upper bound can be estimated as: $H_h = 1$ (for that it is sufficient to calculate a_n for $n = 10$, 100 a 1000). We test whether this bound hold for any a_n:

$$\frac{n-1}{n+3} < 1$$
$$n-1 < n+3$$
$$-1 < 3$$

this is valid for each $n \in \mathbb{N}$. Therefore the sequence $\left\{\frac{n-1}{n+3}\right\}_{n=1}^{\infty}$ is increasing and bounded:

$$H_l = 0 \text{ a } H_u = 1$$

6. The limit $\lim_{n\to\infty} \frac{4n^3-2n+5}{3n^4+n^2-2n-3}$ can be calculated with help of known limit: $\lim_{n\to\infty} \frac{1}{n^k} = 0$. The original expression will be converted to this type of limits by multiplying the rational expression by factor: $\frac{1/n^4}{1/n^4}$. We get:

$$\lim_{n\to\infty} \frac{4n^3-2n+5}{3n^4+n^2-2n-3} = \lim_{n\to\infty} \frac{\frac{4}{n}-\frac{2}{n^3}+\frac{5}{n^4}}{3+\frac{1}{n^2}-\frac{2}{n^3}-\frac{3}{n^4}} = \frac{0-0+0}{3+0-0-0} = 0$$

7. To calculate the limit of sequence: $\lim_{n\to\infty}\left(1+\frac{1}{5n^2}\right)^{3n^2+2n}$ we use substitution $5n^2 = m$ where: $n \to \infty \Rightarrow m \to \infty$ and the known limit: $\lim_{m\to\infty}\left(1+\frac{1}{m}\right)^m = e$

$$\lim_{n\to\infty}\left(1+\frac{1}{5n^2}\right)^{3n^2+2n} = \lim_{n\to\infty}\left(1+\frac{1}{5n^2}\right)^{\frac{5n^2}{5n^2}(3n^2+2n)} =$$

$$= \lim_{m\to\infty}\left(1+\frac{1}{m}\right)^{m\cdot\lim_{n\to\infty}\frac{1}{5n^2}(3n^2+2n)} = e^{\lim_{n\to\infty}\frac{3n^2+2n}{5n^2}} = e^{\lim_{n\to\infty}\frac{3+\frac{2}{n}}{5}} = e^{\frac{3+0}{5}} = \sqrt[5]{e^3}$$

8. To calculate the limit of sequence: $\lim_{n\to\infty}\left(1+\frac{2}{n}\right)^{\frac{n+1}{2n+3}}$ we use substitution $n = 2m$, where: $n \to \infty \Rightarrow m \to \infty$ and the known limit:$\lim_{m\to\infty}\left(1+\frac{1}{m}\right)^m = e$.

$$\lim_{n\to\infty}\left(1+\frac{2}{n}\right)^{\frac{n+1}{2n+3}} = \lim_{m\to\infty}\left(1+\frac{1}{m}\right)^{\frac{m}{m}\frac{2m+1}{4m+3}} = e^{\lim_{m\to\infty}\frac{2m+1}{4m^2+3m}} = e^{\lim_{m\to\infty}\frac{\frac{2}{m}+\frac{1}{m^2}}{4+\frac{3}{m}}} =$$
$$e^{\frac{0+0}{4+0}} = e^0 = 1$$

9. The limit of sequence: $\lim_{n\to\infty} 4^{\frac{6n+2}{3n-4}}$ can be calculated as:

$$\lim_{n\to\infty} 4^{\frac{6n+2}{3n-4}} = 4^{\lim_{n\to\infty}\frac{6n+2}{3n-4}} = 4^{\lim_{n\to\infty}\frac{6+\frac{2}{n}}{3-\frac{4}{n}}} = 4^{\frac{6+0}{3-0}} = 4^2 = 16$$

10. Infinite geometric series $\sum_{n=0}^{\infty} 2\left(-\frac{3}{5}\right)^n$ has a quotient q of $\left|-\frac{3}{5}\right| < 1$, therefore the series is convergent. The sum is calculated as:

$$\sum_{n=0}^{\infty} 2\left(-\frac{3}{5}\right)^n = 2 - \frac{6}{5} + \frac{18}{25} - \frac{54}{125} + \dots + 2\left(-\frac{3}{5}\right)^n + \dots = 2\frac{1}{1-q} = 2\frac{1}{1-\left(-\frac{3}{5}\right)} = \frac{2}{\frac{8}{5}} = \frac{5}{4}$$

11. We calculate the D'Alembert criterion of convergence for infinite series $\sum_{n=1}^{\infty} \frac{n+1}{n}\left(\frac{1}{2}\right)^n$ and determine whether the series converges:

$$L = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{(n+1)+1}{n+1}\left(\frac{1}{2}\right)^{n+1}}{\frac{n+1}{n}\left(\frac{1}{2}\right)^n}\right| = \lim_{n\to\infty}\left|\frac{\frac{n+2}{(n+1)\cdot 2^{n+1}}}{\frac{n+1}{n\cdot 2^n}}\right| =$$

$$= \lim_{n\to\infty}\left|\frac{n(n+2)2^n}{(n+1)^2 2^{n+1}}\right| = \frac{1}{2}\cdot\lim_{n\to\infty}\left|\frac{n^2+2n}{n^2+2n+1}\right| = \frac{1}{2}\cdot\lim_{n\to\infty}\left|\frac{1+\frac{2}{n}}{1+\frac{2}{n}+\frac{1}{n^2}}\right| = \frac{1}{2}\cdot\frac{1+0}{1+0+0} = \frac{1}{2} < 1$$

Since $L < 1$ the series converges.

12. Interval of convergence of power series: $\sum_{n=1}^{\infty}\frac{1}{n^2}x^n$ is calculated as:

$$r = \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right| = \lim_{n\to\infty}\left|\frac{\frac{1}{n^2}}{\frac{1}{(n+1)^2}}\right| = \lim_{n\to\infty}\left|\frac{n^2+2n+1}{n^2}\right| = \frac{1+0+0}{1} = 1$$

The power series will converge on the interval: $(-1, 1)$.